ON A CLASS OF FUNCTIONAL EQUATIONS OF MODULAR TYPE

BY RICHARD BELLMAN

RAND CORPORATION, SANTA MONICA, CALIFORNIA

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1. *Introduction.*—The Voronoi functions, or generalized Bessel functions,

$$V_a(x, y) = \int_0^\infty e^{-\pi x^2 s - \pi y^2/s}\, ds/s^a, \tag{1.1}$$

play an important role in analytic number theory in connection with lattice-point problems and other investigations involving zeta functions. First introduced by Voronoi in his classic paper on the Dirichlet divisor problem, they have been treated by Hardy[1] and also by Steen,[2] who considered the more general functions defined by the relation

$$\int_0^\infty V(x; a_1, a_2, \ldots, a_k)x^{s-1}\, dx = \Gamma(s + a_1)\Gamma(s + a_2) \ldots \Gamma(s + a_k), \tag{1.2}$$

a class of functions whose importance was recognized by Voronoi. Extensions of these functions in the directions of algebraic number fields and matric fields have been sketched by Bellman,[3] and the matrix functions have been discussed in some detail by Bochner.[4] The function arises in the research of Hecke and in the continuation by Maass.[5]

In a previous paper[3] we sketched some extensions of the method used by Hardy to obtain a number of striking identities satisfied by series formed with these functions. In this paper our aim is to indicate how to form extensive classes of functions satisfying functional equations of modular type, using the function above and various extensions and generalizations within the rational field, algebraic number fields, matric fields, hypercomplex fields, and finite fields.

A more detailed account will be presented subsequently.

2. *The Rational Field.*—Consider the function defined by the series

$$f(x, y; u, v; t_1, t_2) = \sum_{m, n = -\infty}^{\infty} V_a((x + m)\sqrt{t_1}, (y + n)\sqrt{t_2})e^{2\pi i(mu + mv)}, \tag{2.1}$$

where $0 < x, y, u, v < 1$ and $t_1, t_2 > 0$. Using the representation in equation (1.1), this may be written as

$$f(x, y; u, v; t_1, t_2) = \int_0^\infty \left[\sum_{m=-\infty}^{\infty} e^{-\pi(x+m)^2 s t_1 + 2\pi i m u}\right] \left[\sum_{n=-\infty}^{\infty} e^{-\pi(y+n)^2 t_2/s + 2\pi i n v}\right]\frac{ds}{s^a}. \tag{2.2}$$

Applying the functional equation for the theta function in each of the brackets (in the previous paper[3] this was applied to only one of the brackets), we obtain the functional relation

$$f(x, y; u, v; t_1, t_2) = \frac{e^{-2\pi i(xu+yv)}}{\sqrt{t_1 t_2}} f(v, u; -y, -x; t_2^{-1}, t_1^{-1}).$$

3. *An Observation*,—Let us note that no properties of the weight ds/s^3 have been used. Consequently, the same functional equation holds for any weight function $dG(s)$ permitting the interchanges of summation and integration.

In addition, contour integrals of the form $\int_c e^{-\pi x^2 z - \pi y^2/z} dz/z^a$ may be used, with different contours yielding different functions, and slightly different types of identities.

4. *Specialization.*—If we allow x, y, u, and v to assume rational values, a number of interesting identities involving sums over divisors in specified residue classes are obtained.

This is a consequence of the relation

$$V_a(x, y) = x^{2a-2} \int_0^\infty e^{-\pi s - \pi (xy)^2/s} \frac{ds}{s^a} = x^{2a-2} V_a(|xy|), \qquad (x \neq 0). \quad (4.1)$$

Consider, in particular, the case where $a = 1$, and $x = y = u = v = 1/2$. Then equation (2.2) yields, with $t_1 t_2 = t^2$,

$$\sum_{m,\, n=-\infty}^{\infty} V_a\left(\frac{t|(1+2m)(1+2n)|}{4}\right)(-1)^{m+n} = -\frac{1}{t}\sum_{m,\, n=-\infty}^{\infty} V_a\left(\frac{|(1+2m)(1+2n)|}{4t}\right)(-1)^{m+n} \quad (4.2)$$

or

$$\sum_{R=1}^{\infty} V_a\left(\frac{Rt}{4}\right) a(R) = -\frac{1}{t}\sum_{R=1}^{\infty} V_a\left(\frac{R}{4t}\right) a(R), \quad (4.3)$$

where

$$a(R) = \sum_{|(1+2m)(1+2n)|=R} (-1)^{m+n}, \qquad -\infty < m, n < \infty.$$

Similar identities arise from other sets of rational values for x, y, u, and v.

5. *Generalizations in the Rational Field. I.*—Using the generalized Voronoi function in the form

$$V(x_1, x_2, \ldots, x_{k+1}; a_1, a_2, \ldots, a_k) = \int_0^\infty \cdots \int_0^\infty e^{-\pi s_1 x_1^2 - \ldots - \pi s_k x_k^2 - \pi x_{k+1}^2/s_1 s_2 \ldots s_k} \prod_{i=1}^{k} \frac{ds_i}{s_i^{a_i}} \quad (5.1)$$

cf. Steen[2] and Bellman,[3] a relation analogous to that of equation (2.2) can be obtained for the function defined by the series

$$\sum_{m_i=-\infty}^{\infty} V\left((x_1+m_1)\sqrt{t_1}, \ldots, (x_{k+1}+m_{k+1})\sqrt{t_{k+1}}; a_1, a_2, \ldots, a_k\right) e^{2\pi i \sum_{i=1}^{k+1} m_i u_i}, \quad (5.2)$$

where $0 < x_i, u_i < 1, t_i > 0$. These identities yield analogues of equation (4.3) for higher-order divisor functions.

6. *Generalizations in the Rational Field. II.*—Two further lines of generalization are immediate. In place of the two-dimensional version of equation (5.1), we may employ integrals of the type

$$\int_0^\infty \int_0^\infty e^{-\pi(s_1x_1^2+s_2x_2^2 + s_1s_2x_3^2-x_4^2/s_1s_2]} \frac{ds_1ds_2}{s_1^{a_1}s_2^{a_2}}, \tag{6.1}$$

utilizing all combinations of products, one at a time, two at a time, and so on. In place of forming sums such as that of expression (5.2), we can form sums of the type

$$\sum_{m_1=-\infty}^{\infty} V_a([Q((x_1 + m_1), (x_2 + m_2))]^{1/2}, (y + n) \sqrt{t_4})e^{2\pi i(mu_1+m_2u_2+nu_3)}, \tag{6.2}$$

where $Q(u, v) = t_1u^2 + 2t_2uv + t_3v^2$ is a positive-definite quadratic form. Use of the multidimensional theta-function transformation will yield the desired functional equation.

7. *Algebraic Number Fields.*—Proceeding in the manner made classical by the work of Hecke, it is easy to form the corresponding Voronoi functions in totally real fields. Thus, for example, in $R(\sqrt{2})$, we have, as the analogue of the function in equation (1.1),

$$V_a(x, y) = \int_{s>>0} e^{-tr(\pi x^2s + \pi y^2/s)} \frac{ds_1ds_2}{(s_1^2 - 2s_2^2)^a}, \tag{7.1}$$

where the integration is over the region defined by $s_1 \geq \pm s_2\sqrt{2}$, and $x = x_1 + x_2\sqrt{2}$, $y = y_1 + y_2\sqrt{2}$.

8. *Matric Fields.*—Similarly, over matric fields, following the work of Siegel, we may define the function of two symmetric matrices X and Y,

$$V_a(X, Y) = \int_{S>0} e^{-tr(\pi X^2S+\pi Y^2S^{-1})} \frac{dS}{|S|^a}, \tag{8.1}$$

where the integration is over the region where the symmetric matrix S is positive definite, $dS = \prod_{1 \leq i \leq j \leq n} ds_{ij}$, and $|S|$ is the determinant of S (cf. Bellman[3] and Bochner[4]).

9. *Discussion.*—In a like fashion, generalized Bessel functions can be defined over a number of hypercomplex fields for which theta functions exist. This is an interesting direction of research which we shall develop in future papers.

10. *Finite Fields—Kloostermann Sums.*—An analogue of the Voronoi function in finite fields is the Kloostermann sum

$$K(x, q, p) = \sum_{n=1}^{p-1} e^{-2\pi i(x^2n+n^{-1})q/p}, \tag{10.1}$$

with a variety of other analogues obtainable from the continuous forms mentioned above. Results corresponding to those sketched earlier[3] and to those above can be obtained for these functions.

[1] G. H. Hardy, "Some Multiple Integrals," *Quart. J. Math.*, 39, 357–375, 1908.

[2] S. W. P. Steen, "Divisor Functions: Their Differential Equations and Recurrence Formulae," *Proc. London Math. Soc.*, **31**, 47–80, 1930.

[3] R. Bellman, "Generalized Eisenstein Series and Non-analytic Automorphic Functions," these PROCEEDINGS, **36**, 356–359, 1950. See also R. Bellman, "Wigert's Approximate Functional Equation and the Riemann Zeta-Function," *Duke Math J.*, **16**, 547–552, 1949.

[4] S. Bochner, *Harmonic Analysis and the Theory of Probability* (Berkeley: University of California Press, 1955).

[5] H. Maass, "Über eine neue Art von nichtanalytischen automorphen Funktionen und die Bestimmung Dirchletschen Reihen durch Funktionalgleichungen," *Math. Ann.*, **121**, 141–183, 1949.

ON THE PRINCIPLE OF INVARIANT IMBEDDING AND PROPAGATION THROUGH INHOMOGENEOUS MEDIA

BY RICHARD BELLMAN AND ROBERT KALABA

RAND CORPORATION, SANTA MONICA, CALIFORNIA

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1. *Introduction.*—In 1943, Ambarzumian[1] introduced a new approach to the study of atmospheric scattering problems. This technique was considerably extended by Chandrasekhar,[2] who utilized it to resolve a number of long-standing problems and gave it the name "principle of invariance."

In a series of papers, of which this is the first, we shall show that the full power of the method has not yet been realized. A number of problems of current interest, in scattering theory, in neutron diffusion, in the propagation of radio waves through stochastic media, as well as a variety of problems in other fields, may be treated by its use.

In addition to the varied physical applications, quite a number of mathematical questions arise concerning the relation between this approach and the classical techniques based upon partial differential and integral equations. These questions will be discussed subsequently.

In this note we are primarily interested in methodology. As an illustration of the general method, we shall discuss the scattering of light by an inhomogeneous plane medium of finite thickness. The corresponding problem for homogeneous media was treated by Ambarzumian and Chandrasekhar.

2. *Principle of Invariant Imbedding.*—Although space limitations do not permit us to discuss in any detail the close connections existing between the approach presented below and a number of fundamental techniques of classical analysis, we would like to mention some of these relations in a very brief fashion.

To begin with, the method is closely related to the "point of regeneration" method used in the study of stochastic processes, particularly in the study of "branching processes" arising in the study of bacterial growth, electron cascades, and cosmic-ray cascades (cf. Bellman and Harris;[3] S. Janossy[4]). Second, the invariance principle is intimately related to the principle of causality and hence to Huygens' principle (cf. Hadamard[5]). Consequently, there is a close connection with the theory of semigroups (cf. Hille[6]).